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Quantum Groups, Deformed Oscillators and their Interrelations

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Abstract.

The main notions of the quantum groups: coproduct, action and coaction, representation and corepresentation are discussed using simplest examples: $GL_q(2)$, $sl_q(2)$, q-oscillator algebra $\mathcal{A}(q)$, and reflection equation algebra. The Gauss decompositions of quantum groups and their realizations in terms of $\mathcal{A}(q)$ are given.

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Despite of the intensive and successful development of mathematical theory of quantum groups the physical interpretations (and physical applications) of many results in this field deserve greater attention. In such situation it is helpful to consider some simple examples and their interrelations. The simplest are depicted in the central part of the diagram shown in the Fig.1. In the boxes, drawn at the left and right columns surrounding the biggest central box, we indicate the main sources of deformed objects. The arrows started from these small boxes show to respective deformed structures.

The central box of this diagram has three levels, indicated by dotted boxes. At the highest level we indicate two, well known objects. Namely, the **quantum groups**, $Fun_q G \equiv F_q(G)$ or, more precisely, " *deformations of function algebras $Fun(G)$ of the (simple) Lie groups G* " , and **quantum algebras**, $U_q(g)$ or, more precisely, "*deformations of universal enveloping algebras $U(g)$ of the simple Lie algebras g.* For the objects on this level we have complete theory. In particular they have the reach additional structure – the Hopf algebra structure [1,2], so we called this level – *Hopf algebra level.* We recall that the Hopf algebra characterized by the existence apart from the usual multiplication the additional coalgebraic structure which defined by three maps: **coproduct** Δ; **counity** ϵ and **coinverse** or **antipod** S.

The objects depicted at the middle level are also developed well. They are the so-called **quantum spaces** with noncommuting coordinates including **external quantum algebras** on which coaction of quantum groups and/or quantum algebras are usually given in a manner which resembles in some aspects standard actions of Lie groups and Lie algebras on usual vector spaces with commuting coordinates. On this level we see also such nowadays popular objects as **deformed oscillator algebras** and **reflection equation algebras** [3,4]. For all objects on this level the Hopf algebra structure is not known up to now. So we call this level – **non Hopf algebra level** .

The bottom level of central box we call **Hopf algebra** and/or **not Hopf algebra level**, because for the objects mentioned here the Hopf algebra structure is exist or probably exist but sometimes not known at present.

The arrows connected the internal boxes in the central one indicate the main interrelations between respective structures. In particular, the double arrows connected the boxes, which are situated at the highest (Hopf algebra) level, mean the most known duality connection between quantum groups and quantum algebras somewhat similar to the exponential map connected the Lie algebras with the Lie groups. We would like to mention the so-called q-bosonization, that is expression of generators of deformed objects in terms of creation and annihilation operators which constitute the basis of q-oscillator algebra. We also show by arrows coaction of quantum groups and quantum algebras on quantum spaces, q-oscillator and reflection equation algebras.

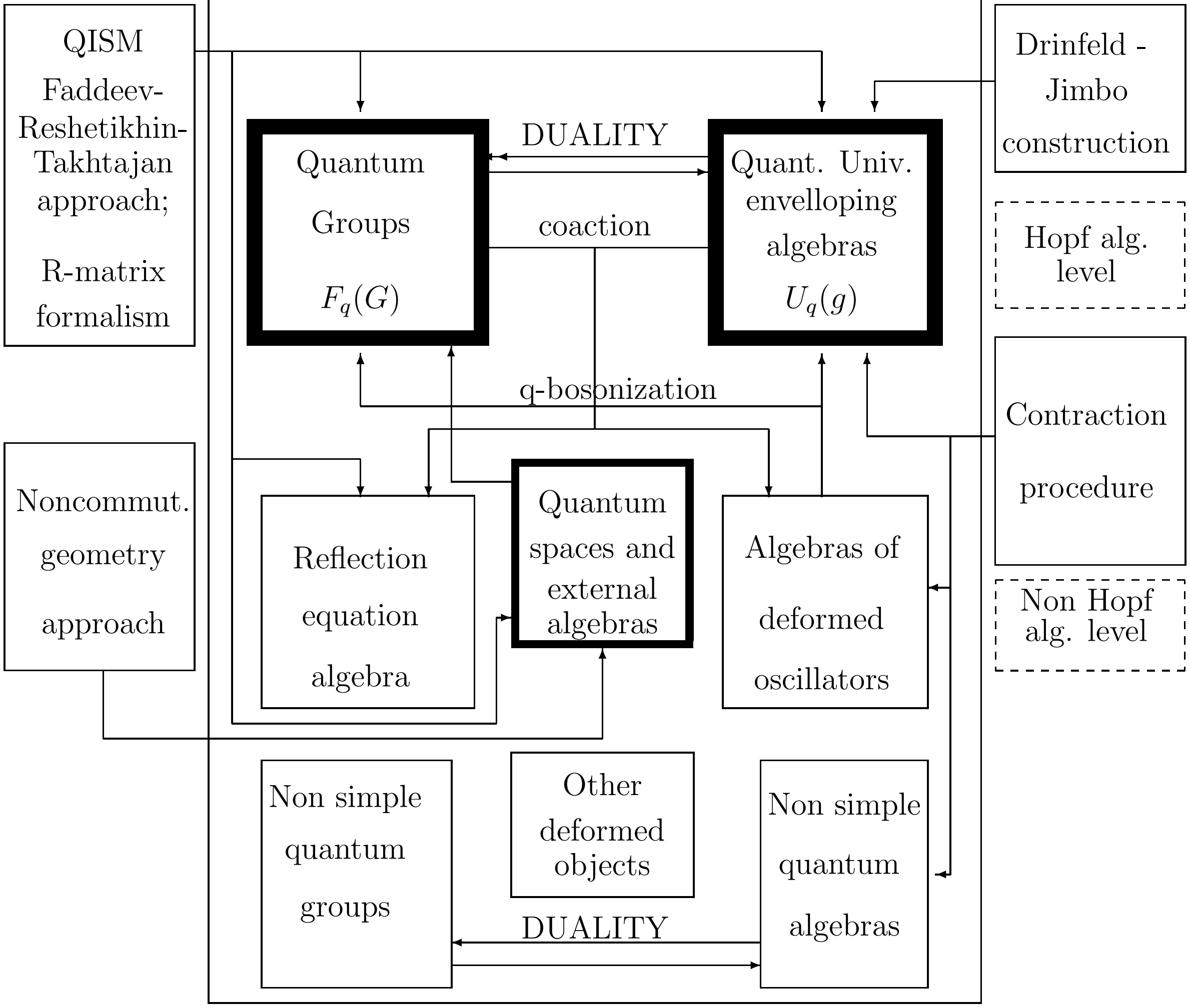

Fig.1 The main quantum deformed objects
and some of their interrelations

Of course, many important relations and interesting deformed objects (such as exchange algebras, Sklyanin algebra and other quadratic deformations, braided structures etc.) are missed in such simple scheme, but we think that it may be helpfull, especially for beginners, in understanding the situation as a whole. We do not dwell here upon interesting problems of representation and corepresentation theory of deformed objects, and, in particular upon astonishing aspects connected with not generic values of deformation parameter q (q is a root of unity). The rich circle of problems connected with q-analysis and q-special functions is also left aside (cf. few contributions to these Proceedings).

The main questions which we want to discuss briefly in our talk are:

1) the using of coaction and action in the definition of covariant objects when the structure of Hopf algebra is absent;

2) the q-bosonization of quantum groups and reflection equation algebras;

3) the Gauss decomposition of quantum matrix.

For simplicity we restrict our considerations to the $n = 2$ case ($GL_q(2), SL_q(2)$, $\mathcal{K}(2)$ etc.) but almost all results can be extended, rather simply, to the general case $n > 2$.

Let us begin with short review of the definition of the quantum groups in the framework of Faddeev-Reshetikhin-Takhtajan (FRT-) or R-matrix approach [1] taking $GL_q(2)$ as example. The R-matrix for this quantum group has the form

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \ \widehat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \ \mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

here $q \in \boldsymbol{C}$, $q \neq 0$, $|q| \neq 1$, $\lambda = q - q^{-1}$, R is the so called R-matrix corresponding to $GL_q(2)$, $\widehat{R} = \mathcal{P}R$ it's braided (modified) form and $\mathcal{P}$ is permutation operator ($\mathcal{P}(a \otimes b) = b \otimes a$). We recall that R is the number matrix which is a solution of the famous Yang–Baxter equation(YB-eq.): $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, where we used standard QISM-notation [1]. For quantum group $GL_q(2)$, and generally for $GL_q(n)$, R-matrix additionally fulfill the Hecke condition $\widehat{R}^2 = \lambda\widehat{R} + I \quad \Leftrightarrow \quad (I - q^{-1}\widehat{R})(I + q\widehat{R}) = 0$.

Let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is the quantum matrix whose entries are the generators of the quantum group $GL_q(2)$.The defining relations of the $GL_q(2)$ generators are encoded in the FRT-relation [1] $RT_1T_2 = T_2T_1R$, where $T_1 := T \otimes 1$, $\ T_2 := 1 \otimes T$, or element wise

$$\begin{aligned} ab = qba, \quad ac = qca, \quad [b, c] = 0, \\ bd = qdb, \quad cd = qdc, \quad [a, d] = \lambda bc. \end{aligned} \tag{2}$$

As an algebra $GL_q(2)$ can be defined as associative $\mathbf{C}$-algebra with unity 1, generated by elements a, b, c and d, subject the relations (2).

It can be easily checked [1] that element $D_q = \det_q T := ad - qbc = da - q^{-1}bc$, commutes with every element of $GL_q(2)$, moreover, it can be proved that the center of $GL_q(2)$ are generated by 1 and quantum determinant D_q. It was generally supposed that $D_q \neq 0$. The additional assumption that $D_q = 1$ defines quantum group $SL_q(2)$. We also note that if ϵ_q denote the q-metric matrix, $\epsilon_q = \left(\begin{smallmatrix} 0 & 1 \\ -q & 0 \end{smallmatrix}\right)$, then we have $T\epsilon_q T^t = T^t\epsilon_q T = \epsilon_q D_q$.

Quantum group $GL_q(2)$ besides the algebraic structure, described above, has an additional coalgebraic structure, which defined by three maps:

$$\begin{aligned} \textbf{coproduct}: \quad \Delta: \quad GL_q(2) \quad &\rightarrow \quad GL_q(2) \otimes GL_q(2); \\ \textbf{counity}: \quad \varepsilon: \quad GL_q(2) \quad &\rightarrow \quad \mathbf{C}; \\ \textbf{coinverce} \ \mathit{or} \ \textbf{antipod}: \quad S: \quad GL_q(2) \quad &\rightarrow \quad GL_q(2). \end{aligned}$$

The first two of them are homeomorphisms and the latest is antihomomorphism

$$\Delta(XY) = \Delta(X)\Delta(Y); \ \ \varepsilon(XY) = \varepsilon(X)\varepsilon(Y), \ \ S(XY) = S(Y)S(X),$$

$\forall X, Y \in GL_q(2)$. This maps defined by the relations

$$\Delta(T) = T\dot{\otimes}T; \quad \Delta(1) = 1\dot{\otimes}1; \quad \Delta(D_q) = D_q\dot{\otimes}D_q; \tag{3}$$

$$\varepsilon(T) = I; \qquad \varepsilon(1) = 1; \tag{4}$$

$$S(T) = T^{-1} = D_q^{-1}\begin{pmatrix} d & -b/q \\ -qc & a \end{pmatrix}; \quad S(1) = 1. \tag{5}$$

There are three real form of quantum group $GL_q(2)$ (namely (i) $U_q(2)$, (ii) $U_q(1,1)$ and (iii) $GL_q(2;\mathbf{R})$) corresponding to three possible types of involution. We will use two of them ($q \in \mathbf{R}$, $\overline{q} = q$):

$$U_q(2): \;\; T^\dagger = D_q^{-1}\begin{pmatrix} d & -qc \\ -b/q & a \end{pmatrix}; \qquad U_q(1,1): \;\; T^\dagger = D_q^{-1}\begin{pmatrix} d & qc \\ b/q & a \end{pmatrix}$$

If moreover $D_q = 1$, then we received quantum groups $SU_q(2)$, $SU_q(1,1)$ and $SL_q(2;\mathbf{R})$, respectively.

As the simplest example of the dual type of deformed objects – quantum algebras [1,2], consider quantum deformation $U_q sl(2) \equiv sl_q(2)$ of universal enveloping algebra $Usl(2)$ of Lie algebra $sl(2)$. It is an associative algebra with unity generated by three elements $J, X_\pm$ subject the commutation relations

$$[J, X_\pm] = \pm X_\pm, \qquad [X_+, X_-] = [2J], \tag{6}$$

here $[a] := \frac{q^a - q^{-a}}{q - q^{-1}}$. This algebra also has the nontrivial center, generated by the q-analog $c_2(q) = X_-X_+ + [J][J+1] = X_+X_- + [J][J-1]$ of the well-known Casimir element of $sl(2)$ Lie algebra. As quantum algebra $sl_q(2)$ has also the Hopf algebra structure:

$$\Delta(J) = J \otimes I + I \otimes J; \quad \Delta(X_\pm) = X_\pm \otimes q^{-J} + q^J \otimes X_\pm; \quad \Delta(I) = I \otimes I; \tag{7}$$

$$\varepsilon(J) = \varepsilon(X_\pm) = 0; \quad \varepsilon(I) = 1; \tag{8}$$

$$S(J) = -J; \quad S(X_\pm) = -q^\mp X_\pm; \quad S(I) = I; \tag{9}$$

In R-matrix approach commutation relations (6) are described [1] by three equations $R^\pm L_1^\pm L_2^\epsilon = L_2^\epsilon L_2^\epsilon R^\pm$, $\epsilon = +, -$. Here $R^+ = q^{-1/2}\mathcal{P}R\mathcal{P}$, $R^- = q^{1/2}R^{-1}$ and L^+ and L^- are upper and lover triangular 2×2-matrices given bellow

$$L^+ = \begin{pmatrix} q^J & \lambda X_- \\ 0 & q^{-J} \end{pmatrix} \quad L^- = \begin{pmatrix} q^{-J} & 0 \\ -\lambda X_+ & q^J \end{pmatrix} \tag{10}$$

The above mentioned duality pairing between quantum groups and quantum algebras in the considered case of quantum group $GL_q(2)$ and quantum algebra $sl_q(2)$ established [1] by the relations $\langle L_1^{\pm}, T_2\rangle = R_{12}^{\pm}, \quad \langle L_1^{\pm}, T_2T_3\rangle = R_{12}^{\pm}R_{13}^{\pm}, \ldots$.

As $GL_q(2)$ the quantum algebra $sl_q(2)$ also has three real forms: denoted as $su_q(2)$, $su_q(1,1)$ and $sl_q(2,\mathbf{R})$, corresponding to the related real forms of $GL_q(2)$.

For the general value of q (q is not a root of unity) the representation theory for quantum algebras $sl_q(n)$ looks much the same as for the $sl(n)$ Lie algebras. In particular for the $su_q(2)$ we have the infinite series of finite-dimensional irreducible representations $\mathcal{V}_n$, $n = 0, 1/2, 1, 3/2, \ldots$, $\dim\mathcal{V}_n = 2n+1$. (see for example [5]).

Lie group $GL(2)$ can be considered as group of endomorphismes 2-dimensional linear space, so it is natural to seek the similar object for quantum group $GL_q(2)$ also. Such objects indeed can be defined [1]. The simplest one is the so-called quantum plane $\mathbf{C}_q^{[2]}$. It is an associative algebra generated by to generators x_1, x_2 which can be considered as non-commuting coordinates of quantum vector $X = \binom{x_1}{x_2}$. It supposed that this coordinates have the following simple commutation rule $x_1x_2 = qx_2x_1$. There is another related simple object [1] – **external algebra of quantum plane** or **Grassmann quantum plane**, which is in the some sense dual to quantum plane. This Grassmann q-plane also can be defined as associative algebra $\mathbf{C}_q^{0|2}$ with two generators ξ_1, ξ_2 - coordinates of the "vector" $\Xi = \binom{\xi_1}{\xi_2}$, with relations $\xi_1\xi_2 = q\xi_2\xi_1, \quad (\xi_1)^2 = (\xi_2)^2 = 0$. This commutation rules for $\mathbf{C}_q^{[2]}$ and $\mathbf{C}_q^{0|2}$ can be written in the elegant R-matrix form [1] as

$$\mathbf{C}_q^{[2]}: \ \ \widehat{R}\ X \otimes X = qX \otimes X; \qquad \mathbf{C}_q^{[0|2]}: \ \ \widehat{R}\ \Xi \otimes \Xi = -q^{-1}\Xi \otimes \Xi. \tag{11}$$

The relations (11) are particular cases of the relation [1] $f(\widehat{R})\,(X \otimes X) = 0$,where $f(t)$ arbitrary polynomial. The considered above cases of q-plane and Grassmann q-plane are obtained when we take $f(t) = t - q$ and $f(t) = t + q^{-1}$, respectively. We remarks that the relations between coordinates of Ξ are thus the relations which are expected for differentials of non commuting coordinates on q-plane, $\xi_i = dx_i$, but we not dwell on this subject here. We also note that conditions $X \in \mathbf{C}_q^{[2]} \iff (TX) \in \mathbf{C}_q^{[2]}$ and $\Xi \in \mathbf{C}_q^{[0|2]} \iff (T\Xi) \in \mathbf{C}_q^{[0|2]}$, where T is a 2×2-matrix, considered together are sufficient to define the q -commutation relations of quantum group $GL_q(2)$.

We can define $GL_q(2)$-coactions of quantum group $GL_q(2)$ on $\mathbf{C}_q^2$ and $\mathbf{C}_q^{[0|2]}$, respectively, by

$$\varphi : \mathbf{C}_q^2 \to GL_q(2) \otimes \mathbf{C}_q^2 : X \to X_T = T\dot{\otimes}X; \tag{12}$$

$$\varphi^* : \mathbf{C}_q^{[0|2]} \to GL_q(2) \otimes \mathbf{C}_q^{[0|2]} : \Xi \to \Xi_T = T\dot{\otimes}\Xi. \tag{13}$$

This $GL_q(2)$-coactions are consistent with coproduct and counity in $GL_q(2)$, that is relations

$$(\Delta \otimes id) \circ \varphi = (id \otimes \varphi) \circ \varphi, \qquad (\epsilon \otimes id) \circ \varphi = id, \tag{14}$$

and similar relations for φ^* are fulfilled. Thus $\mathbf{C}_q^{[2]}$ and $\mathbf{C}_q^{[0|2]}$ may be considered as $GL_q(2)$ -comodules.

The simplest quantum deformed object is a q-deformed oscillator algebra [5-7] $\mathcal{A}(q)$ which is, may be physically most interesting one. As above $\mathcal{A}(q)$ is associative algebra with three generators $a, a^\dagger$ and N subject to the following commutation relations

$$aa^\dagger - qa^\dagger a = q^{-N}, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger. \tag{15}$$

Let us stress that here N is a generator completely independent of $a, a^\dagger$.

The Fock representation $\mathcal{H}$ of this algebra $\mathcal{A}(q)$ can be easily constructed. Let us define the q-vacuum state by the natural relations $a_F|0\rangle_F = 0$; $N_F|0\rangle_F = 0$. Then the states $|n\rangle_F = ([n]!)^{-1/2}(a^\dagger)^n|0\rangle_F$, where $[x] := \frac{q^x - q^{-x}}{q - q^{-1}}$ forms the complete basis in Fock space $\mathcal{H}_F$. In this Fock space the action of the generators of q -oscillator algebra $\mathcal{A}(q)$ are given by: $N|n\rangle_F = n|n\rangle_F$;

$$a_F|n\rangle_F = (1 - \delta_{n,0})\sqrt{[n]}|n-1\rangle_F; a_F^\dagger|n\rangle_F = \sqrt{[n+1]}|n+1\rangle_F. \tag{16}$$

But in contrast with the case of the usual oscillator for which the famous von Neumann uniqueness theorem is hold, in the deformed case there are many other representations of $\mathcal{A}(q)$ non equivalent with the Fock one [2,8]. The reason for such difference consist in that the algebra $\mathcal{A}(q)$ has nontrivial center generated by element $c_q = q^{-N}(a^\dagger a - [N])$, which takes in the Fock representation the zero value, because in the Fock case some additional relations

$$a_F a_F^\dagger - q^{-1}a_F^\dagger a_F = q^{N_F}, \;\; a_F^\dagger a_F = [N_F], \;\; a_F a_F^\dagger = [N_F], \tag{17}$$

are hold which are absent in all other representations. Moreover in the Fock representation we have direct connection between usual non deformed operators $b, b^\dagger$ and $N_b = b^\dagger b$ and deformed ones. This connection is given by [5]

$$N_F = N_b, \;\; a_F^\dagger = \sqrt{[N_F]/N_F}\, b^\dagger, \;\; a_F = \sqrt{[N_F + 1]/(N_F + 1)}\, b.$$

Let us stress once more that for the algebra $\mathcal{A}(q)$ the Hopf algebra structure is at least not known and most probably absent.

We note that there are some other variants of the definition of q -oscillator algebra different from given above in some details. It is worthwhile to give some remarks about most popular of them.

Some authors are preferred to use the restricted form $\mathcal{A}(q, q^{-1})$ of the algebra $\mathcal{A}(q)$, in which *ab initio* supposed that extended list of commutation relations are hold. Namely

besides the relations (15) the first of the relations (17) are taken as defining relations. In this case center is trivial $c_q = 0$, and, respectively this algebra has only one up to equivalence irreducible representation – the Fock representation, described above. Provided that the above mentioned map connected the operators of deformed q-oscillator with usual one is invertible (this is not the case when $q^M = 1,\ M \in\mathbf{N}$ this restricted algebra $\mathcal{A}(q,q^{-1})$ is equivalent with standard quantum mechanical boson oscillator algebra. On the other hand this algebra $\mathcal{A}(q,q^{-1})$ can be identified with $sl_q(2)$ quantum algebra. Indeed if both relations $aa^\dagger - qa^\dagger a = q^{-N},\ \ aa^\dagger - q^{-1}a^\dagger a = q^N$ are valid, then operators $X_+ = \sigma a,\ X_- = \sigma a^\dagger,\ J = 1/2(N - \frac{\pi i}{2\eta}),\ (q = e^\eta)$, where $\sigma^2 = \frac{i\sqrt{q}}{q-1}$, fulfill commutation relations of $sl_q(2)$. This equivalence of course allows one to induce the Hopf algebra structure in $\mathcal{A}(q,q^{-1})$ from $sl_q(2)$, but corresponding coproduct does not respect Hermitian conjugation. Another coproduct can be introduced for $\mathcal{A}$ using its isomorphism to $SU_q(2)$ (provided $c_q = 0$).

The another example of q-oscillator algebra gives the algebra $\mathcal{A}(q;\alpha)$, defined by commutation rules $[\alpha,\alpha^\dagger] = q^{-2N},\quad [N,\alpha] = -\alpha,\quad [N,\alpha^\dagger] = \alpha^\dagger$. This algebra also has nontrivial center generated by unity and operator $\zeta_q = \alpha^\dagger\alpha - [N;q^{-2}]$, where $[x;q] = \frac{q^x-1}{q-1}$. So this algebra also has a rich representation theory and in particular Fock representation which may be constructed along the same lines as above. The generators of this algebra related with generators of $\mathcal{A}(q)$ by the relations $a = q^{N/2}\alpha,\ \ a^\dagger = \alpha^\dagger q^{N/2}$.

Our last example is the oldest one. It appear [9,10] approximately ten years before the quantum groups. The related associative algebra $\mathcal{A}(q;A)$ has only two generators A, and $A^\dagger$ and one relation

$$AA^\dagger - qA^\dagger A = 1 \tag{18}$$

Of course we may add the number operator N with the standard relations, but absence of N in eq.(18) says that this procedure is independent and not necessary. As in the preceding case the generators of this algebra $\mathcal{A}(q;A)$ can be constructed from the generators of $\mathcal{A}(q)$ by simple formulae. For example operators [2] $\widehat{A} = q^{N/2}a,\ \widehat{A}^\dagger = a^\dagger q^{N/2}$ fulfil the relation $\widehat{A}\widehat{A}^\dagger - q^2\widehat{A}^\dagger\widehat{A} = 1$ of the same type as (18) but with squared deformation parameter. Let us note that the relations (18) can be put into the R-matrix form

$$\widehat{R}(X\otimes X) = q(X\otimes X) + V,\quad V^t = (0,-1/q,1,0),\quad X = \begin{pmatrix} A \\ A^\dagger \end{pmatrix}.$$

Note that $\mathcal{A}(q;A)$ is also a $SU_q(1,1)$-comodule algebra under the coaction

$$\psi : X \mapsto \widehat{X} = TX = \begin{pmatrix} aA + bA^\dagger \\ b^*A + a^*A^\dagger \end{pmatrix};\quad T = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \in SU_q(1,1).$$

We note also that there is non trivial generalization to the $SU_q(n)$ -covariant algebra for the case of $n \geq 2$ q-oscillators [11], in which the covariance condition dictates the type of commutation relations between different q-oscillators. (For the SUSY case see [12]).

We remark in conclusion of our brief list of some properties of different types of q-oscillator algebras, that among different types of their representations we can find specific ones which are not survive in the "classical" limit $q \to 0$ [8]. As example of such singular representation for the algebra $\mathcal{A}(q;A)$ we may consider the following representation $A\Psi_n = \beta^{-1/2}q^{-1/4}\Psi_{n-1}$, $A^\dagger\Psi_n = \beta^{-1/2}q^{-1/4}\Psi_{n+1}$, where $\beta = q^{-1/2} - q^{1/2}$, for $0 < q < 1$.

The last type of the quantum deformed objects which we want to discuss briefly is the reflection equation algebra $\mathcal{K}$ [3,4,13]. This is also associative algebra generators of which fulfill commutation rules encoded by reflection equation $RK_1R^{t_1}K_2 = K_2R^{t_1}K_1R$, which has slightly more complicated form compared with FRT-relation given above. In this equation K is the matrix formed by generators, $K_1 = K \otimes I$, $K_2 = I \otimes K$ and R^{t_1} denotes the usual R-matrix R transposed in first space, that is if $R = \sum A_i \otimes B_i$ then $R^{t_1} = \sum A_i^t \otimes B_i$.

It is well-known that quantum group equation appeared in QISM in the course of description of quantum scattering along the axis. Analogously the reflection equation appears similarly when ones considered the scattering processes on half axis. We remark that sometimes one considers little bit different forms of reflection equation but here we restrict ourself to the form given above.

For the case $n = 2$ the commutation relations for generators, considered as elements of a matrix $K = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, received from reflection equation are

$$\begin{array}{lll} [\alpha,\beta] = \lambda\alpha\gamma, & [\alpha,\delta] = \lambda(q\beta+\gamma)\gamma, & \alpha\gamma = q^2\gamma\alpha, \\ [\beta,\delta] = \lambda\gamma\delta, & [\beta,\gamma] = 0, & \gamma\delta = q^2\delta\gamma, \end{array} \tag{19}$$

where as usual $\lambda = q - q^{-1}$. Let us denote $\mathcal{K} = \mathcal{K}(2)$ the received reflection equation algebra. The center of $\mathcal{K}(2)$ is generated by two elements $c_1 = \beta - q\gamma \equiv \mathrm{tr}_q\, K$, $\quad c_2 = \alpha\delta - q^2\beta\gamma \equiv \det_q K$, where $\mathrm{tr}_q\, K = \mathrm{tr}\epsilon_q^t\, K$, $\epsilon_q = \left(\begin{smallmatrix}0 & 1\\ -q & 0\end{smallmatrix}\right)$.

As in the case of q-plane, the Hopf algebra structure for $\mathcal{K}$ is at least not known, but it is $GL_q(2)$–comodule algebra with respect to the coaction of $GL_q(2)$ defined by $\varphi : \mathcal{K} \to GL_q(2) \otimes \mathcal{K} \;:\; K \to \varphi(K) = TKT^t$. So for example $\varphi(\beta) = ac\alpha + ad\beta + bc\gamma + bd\delta$ and $\varphi(c_1) = c_1\, \det_q T$, $\varphi(c_2) = c_2\, (\det_q T)^2$. By $GL_q(2) \leftrightarrow sl_q(2)$ duality $\mathcal{K}$ is also $sl_q(2)$–comodule algebra under $sl_q(2)$–action $\varphi^*(L_1^\pm) : K_2 \to R_{12}^\pm K_2 (R_{12}^\pm)^{t_2}$. As $GL_q(2)$ and $sl_q(2)$, $\mathcal{K}(2)$ has three real form [3]. We remark that $\varphi(K^*) = \varphi(K)^*$. Also note that condition $c_1 = \beta - q\gamma = 0$ defines 2-ideal in $\mathcal{K}(2)$. Under this condition the commutation relations for $\mathcal{K}(2)$ became the commutation rules for generators of quantum group $GL_{q^2}(2)$ with squared deformation parameter. We also remark that this relations are also fulfilled by $K = TT^t$ obtained from $K_T = TK_IT^t$ with $K_I = I$. For $n = 2$ there are two special constant solutions of reflection equation (27) given by matrices $K_0 = \epsilon_q = \left(\begin{smallmatrix}0 & 1\\ -q & 0\end{smallmatrix}\right)$; $\quad K_1 = \left(\begin{smallmatrix}\rho & \mu\\ 0 & \nu\end{smallmatrix}\right)$. Let us recall in conclusion of this brief review of some properties of reflection equation algebra, that it finds the interesting

applications in construction of q-Minkowski spaces and the corresponding non-commutative calculi [26].

Above in consideration of different types of deformed objects we pointed whether this object is supplied by the Hopf algebra structure or not.

But why we are so interested in this structure? Let us recall that in any Hopf algebra $\mathcal{H}$ there are two basic operations: the product m and co-product Δ. This allows one to consider the representations of Hopf algebra $\mathcal{H}$ in some linear or Hilbert space in which m correspond to the usual product of operators. Then the presence of Δ makes it possible to construct the tensor products of different representations of $\mathcal{H}$. Moreover we can consider also a co-representations of Hopf algebra $\mathcal{H}$ for which the basic operation, described by multiplication of operators, is a coproduct Δ. What then we still can do if the coproduct is fail to exist? We would like to note that in such cases we can use the coaction to substitute in a some sense the absent comultiplication operation.

As a simple example we may return to consideration of the quantum group $GL_q(2)$ and the q-plane $\mathcal{V} = \mathbf{C}_q^{[2]}$. As we remarks above the q -plane $\mathcal{V} = \mathbf{C}_q^{[2]}$ is not a Hopf algebra but there is well defined $GL_q(2)$-coaction $\varphi : \mathcal{V} \to GL_q(2) \otimes \mathcal{V}$ such that $\varphi : \binom{x}{y} \mapsto T\binom{x}{y}$, consistent not only with commutation relations in $\mathbf{C}_q^{[2]}$ but with Hopf algebra structure on $GL_q(2)$ also. The last assertion is guaranteed by the conditions (14) which validity are easily to verify in considered case. The validity of this conditions means that $\mathcal{V}$ is $GL_q(2)$ –comodule or $GL_q(2)$–covariant algebra. Now let we have two q-planes $\mathcal{V}_1$ with elements X= $\binom{x}{y}$ and another q-plane $\mathcal{V}_2$ with elements V= $(uopv)$. Than their union will also be a covariant algebra only if the coordinates on different q-planes are subject to special commutation relations, also preserved under coaction. Namely, in considered case together with standard q-commutations $xy = qyx\,, uv = qvu$ additional relations

$$xu = qux\,,\ yv = qvy\,,\ yu = quy\,,\ [x, v] = \lambda uv\,,$$

holds. This relations means that the q-vectors X and V can be considered as columns of $GL_q(2)$-matrix $T = \begin{pmatrix} x & u \\ y & v \end{pmatrix}$. In R-matrix language the above additional relations looks as $R\binom{u}{v} \otimes \binom{x}{y} = \binom{x}{y} \otimes \binom{u}{v}$.

Thus: **Coproduct** Δ defines the action of Hopf algebra $\mathcal{H}$ in tensor product of its representations $\mathcal{V}(\mathcal{H}) \otimes \mathcal{V}(\mathcal{H}) = \sum\{\mathrm{IrReps}(\mathcal{H})\}$.

Coaction φ defines the representation of $\mathcal{H}$–comodule algebra $\mathcal{W}$ in tensor product of representation of $\mathcal{H}$ with representation of $\mathcal{W}$:$\mathcal{V}(\mathcal{H}) \otimes \mathcal{V}(\mathcal{W}) = \sum\{\mathrm{IrReps}(\mathcal{W})\}$.

The contraction procedure may be considered as powerful method of construction of new deformed objects from known ones. In particular such procedure allows us to obtain

q–oscillator algebra $\mathcal{A}(q;\alpha)$, for example from quantum algebra $sl_q(2)$ [5,8,14] :

$$\alpha = \lim_{?\to 0}(?\lambda^{1/2}X_+)\,; \quad \alpha^\dagger = \lim_{?\to 0}(?\lambda^{1/2}X_-)\,; \quad q^{-N} = \lim_{?\to 0}(?q^{-J}).$$

The central element ζ for $\mathcal{A}(q;\alpha)$ also obtainable via such contraction limit $\zeta + q^2/(q^2-1) = \lim_{?\to 0}(?^2\lambda c_2)$. But as we mentioned above the Hopf algebra structure of quantum algebra $sl_q(2)$ does not survive under such contraction limit in this concrete example. Namely Δ goes to $\Psi(\alpha) = \alpha\otimes q^{-J} + \lambda^{1/2}q^{-N}\otimes X_+$; $\Psi(N) = N\otimes 1 - 1\otimes J$. So one can interpret $\Psi : \mathcal{A}(q;\alpha)\otimes sl_q(2)$ as $sl_q(2)$–coaction and $\mathcal{A}(q;\alpha)$ became a $sl_q(2)$ –comodule. So Δ defines the addition of q-spines, whereas Ψ gives the composition of q-spin and q-oscillator that reproduces the q -oscillator algebra $\mathcal{A}(q;\alpha)$.

The last subject which we want touch on in our talk is about of the so called the procedure of q-bosonization that is description of generators of deformed object in terms of creation-destruction operators of the family of quantum oscillators (independent or not). For the quantum algebras such problem was solved rather completely. Let us give some simple examples of different types for $su_q(2)$ quantum algebra. One of them will be already given above: $X_\pm = \sqrt{i\sqrt{q}/(1-q)}a_\pm$. We recall also the most known q-Schwinger representation by two independent quantum oscillators $X_+ = a_1^\dagger a_2$, $X_- = a_2^\dagger a_1$, $J = 1/2(N_1 - N_2)$.

The q-bosonization of noncompact form $su_q(1,1)$ of this quantum algebra can also be given [15]. Let us recall that in this noncompact case the generators are fulfill the commutation rules of the form $[K_0, K_\pm] = \pm K_\pm$, $[K_+, K_-] = -2[K_0]$, and the Casimir operator has the form $c^{[su_q(1,1)]} = [K_0 - 1/2]^2 - K_+K_-$. The most natural q-bosonizations of $su_q(1,1)$ are [15]:

1) One q-boson realization $K_+ = \beta(a^\dagger)^2$, $K_- = \beta a_-^2$, $K_0 = 1/2(N + 1/2)$, where $\beta = (q^{1/2} + q^{-1/2})^{-1}$.

2) Two q-boson or Schwinger-type realization $X_+ = (K_-)^\dagger = a_1^\dagger a_2^\dagger$, $\;K_0 = 1/2(N_1 + N_2 + 1)$.

Let us note that mainly all this q-bosonizations of $sl_q(2)$ hold on the representation level, that is the deformed commutations reproduced in concrete representations, not purely algebraically.

The similar processes of q-bosonization of quantum groups also can be realized (see [16-18] for first attempts in the $GL_q(n)$ case). Here we describe essential steps of the new method suggested recently [19,20]. We take the case of quantum group $GL_q(2)$ as simple example but we would like to stress that this method works for **all series** A_n, B_n, C_n and D_n of 'simple' quantum groups from Cartan list and for **every value** of the range of respective quantum groups. To apply this method we must firstly consider the Gauss decomposition of related q-matrix T, which allows to receive the new set of generators with more simple

commutation rules [19-23]. For the $GL_q(2)$ case this decomposition has the form

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = T_L T_D T_R = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

We can connect the 'new' and 'old' generators by the relations

$$a = A + uBz,\ b = uB,\ c = Bz,\ d = B;$$
$$B = d,\ z = d^{-1}c,\ u = bd^{-1},\ A = a - bd^{-1}c\,,$$

provided that d^{-1} exist. For the quantum determinant we received the expression $\det_q T = \det T_D = AB$. The commutation relations of 'new' generators are

$$AB = BA\,,\ Au = quA\,,\ uB = qBu\,,\ uz = zu\,,\ zB = qBz\,,\ zA = qAz\,.$$

Let us note few main properties of the Gauss decomposition

1) $\widetilde{T} = T_D T_R\,,\ \widehat{T} = T_L T_D Q$ — are new quantum groups with commutation relations and Hopf algebra structure defined by FRT-method with the $GL_q(2)$ –R-matrix (in our case).

2) In $\widetilde{T}$ and $\widehat{T}$ there is another Hopf algebra structure inherited from initial quantum group.

3) $\widetilde{T}$ and $\widehat{T}$ can be connected by duality with Borel subalgebras of dual quantum algebra (in our case $sl_q(2)$).

More simple structure of 'new' generators allows q-bosonization of them in a more simple way. Let us consider two independent family of mutually commuting deformed oscillators. Let first of them consists from q -oscillators and the second from q^{-1}-oscillators: $a_i {a_i}^\dagger - q{a_i}^\dagger a_i = q^{-M_i}\,,\ \ b_i {b_i}^\dagger - q^{-1}{b_i}^\dagger b_i = q^{N_i}$. As examples of the various possible q-bosonizations we cite the following four cases.

1) $u = \lambda q^{-1} {b_1}^\dagger b_2\,,\ z = -q\lambda q^{-1} {a_1}^\dagger a_2\,,\ A = q^{N_1 - M_1}\,,\ B = q^{N_2 - M_2}$.

2) $u = \alpha {a_1}^\dagger, z = \beta a_2\,, A = \gamma q^{M_1 - M_2}\,, B = \delta q^{M_2 - M_1}$, here $\alpha\,\beta\,\gamma\,,\delta$ – arbitrary numbers.

3) $u = \alpha {a_1}^\dagger\,, z = \beta a_2\,, A = \gamma X_1 Y_2\,, B = \delta Y_1 X_2\,,$ where $X_i = \lambda {a_i}^\dagger a_i + q^{-M_i}\,,\ Y_i = \lambda {a_1}^\dagger a_2 - q^{M_i+1}$.

4) $u = \mu q^M W^{-1} a\,, z = \nu W^{-1} q^{M+1} a\,, A = (\lambda q^{M+1} W^{-1}) Q^M a\,, B = a^\dagger\,,$ where $W = q a^\dagger a + q^{-M}$.

As last example we consider the problem of q-bosonization of the reflection equation algebra described above. Unfortunately the Gauss decomposition of the matrix K does not help here because the obtained by this way 'new' generators have more complicated form in comparison with the case of quantum algebras. But there are so-called constant solution of the reflection equation [3,4,13]. If we take such constant solution K and already q-bosonized q-matrix T for $GL_q(2)$ and use the formula $K_T = TKT^t$ we obtain the example of q -bosonization of reflection equation algebra (see also [24,25]).

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